ON THE EQUATION $\sum_{j=1}^{k} jF_j^p = F_n^q$

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Abstract. In this paper, we show that the title equation, where F_m is the mth Fibonacci number, in positive integers (k, n, p, q) with $k > 1$ entails $\max\{k, n, p, q\} \leq 10^{2500}$.

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1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and the recurrence $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. A number of recent papers have studied exponential diophantine equations involving Fibonacci numbers. For example, the equation $F_n^x + F_{n+1}^x = F_m^y$ in positive integers n, m, x, y has been studied in [5], [6] and [8]. The equation $F_n^x + \cdots + F_{n+k-1}^x = F_m$ in positive integers n, k, m, x has been studied in [7] while the equation $F_1^k + \cdots + F_{n-1}^k = F_{n+1}^\ell + \cdots + F_{n+r}^\ell$ in positive integers n, r, k, ℓ has been studied in [2] and [3]. For all these equations, all their positive integer solutions are now known. The title equation has been first studied, to our knowledge, in the paper [10]. Since $F_1 = F_2 = 1$, the title equation has the solutions $(k, n, p, q) = (1, 1, p, q),\ (1, 2, p, q)$ (for any p and q), as well as $(2, 4, p, 1)$ for any p. We call such solutions *trivial*. From now on, we assume that $k \geq 3$. In that paper the authors suggested the following conjecture:

Conjecture 1.1. *The only nontrivial solutions of the title equation are given by* $(k, n, p, q) = (3, 4, 1, 2),\ (3, 4, 3, 3),\ (4, 8, 1, 1)$.

In [10], the authors found all solutions when $\{p, q\} \subseteq \{1, 2\}$ by ad-hoc methods. A general method to find all solutions of the title equation when p and q are given was proposed in [4]. As an application, the authors of [4] confirmed Conjecture 1.1 for the range of the exponents $\{p, q\} \subset [1, 10]$. None of the above papers addressed the question of whether the title equation has only finitely many positive integers solutions and whether the current methods allow us to bound them or even compute them. The aim of our paper is to prove that there are only finitely many effectively computable solutions. Our concrete result is the following.

Theorem 1.1. *The title equation has only finitely many positive integer solutions* (k, n, p, q) *with* $k \geq 3$. *They all satisfy* $\max\{k, n, p, q\} \leq 10^{2500}$.

We made no efforts to try to reduce the above bound. In the last section of the paper we explain why we believe that the computations are not feasible and that new ideas will be required to confirm computationally Conjecture 1.1.

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2. The tools

Most arguments on exponential diophantine equations go via some linear forms in logarithms of algebraic numbers. We are no exception to this. So, let us recall the terminology and the results we need.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in this paper without reference:

$$\begin{array}{rcl} h(\eta\gamma^{\pm 1}) & \leq & h(\eta) + h(\gamma), \\ h(\eta + \gamma) & \leq & h(\eta) + h(\gamma) + \log 2, \\ h(\eta^s) & = & |s|h(\eta) \qquad (s \in \mathbb{Z}). \end{array} \tag{1}$$

In order to prove our main result Theorem 1.1, we need to use six times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such bounds the literature like that of Baker and Wüstholz from [1]. We use the following one of Matveev from [9].

Theorem 2.1 (Matveev's theorem)**.** *Let $\gamma_1, \ldots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1, \tag{2}$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}, \qquad \textit{for all} \qquad i = 1, \ldots, t.$$

Above and anywhere else in the paper, log stands for the natural logarithm.

3. The set up

Recall that the equation is

$$\sum_{j=1}^{k} jF_j^p = F_n^q. \tag{3}$$

As we already mentioned, we take $k \geq 3$. Thus, $n \geq 3$ also holds. Put

$$M := \sum_{j=1}^{k} jF_j^p = F_n^q \qquad \text{and} \qquad X := \log M. \tag{4}$$

We want to bound X. We write

$$(\alpha, \beta) := \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

for the roots of the characteristic equation $x^2 - x - 1$ of the Fibonacci sequence. Then it is well-known that

$$F_\ell = \frac{\alpha^\ell - \beta^\ell}{\alpha - \beta} \quad \text{holds for all} \quad \ell \geq 0.$$

This is sometimes referred to as the *Binet* formula. We also use the fact that the inequalities

$$\alpha^{\ell-2} \leq F_\ell \leq \alpha^{\ell-1} \tag{5}$$

hold for all $\ell \geq 1$. At one point we will need the companion Lucas sequence $\{L_\ell\}_{\ell \geq 0}$ given by $L_0 = 2,\ L_1 = 1$ and $L_{\ell+2} = L_{\ell+1} + L_\ell$ for all $\ell \geq 0$. It satisfies

$$L_\ell < 2^\ell$$

for all $\ell \geq 1$.

4. The sizes of kp versus nq

Lemma 4.1. *We have*

$$10^{-1}X < kp < 10X \quad \textit{and} \quad 10^{-1}X < nq < 10X. \tag{6}$$

Proof. We have

$$\begin{aligned} (\alpha^{k/3})^p &\leq (\alpha^{k-2})^p < F_k^p < M; \\ M &< k(\sum_{j=1}^{k} F_j)^p < k(F_{k+2})^p < k(\alpha^{k+1})^p < \alpha^{(k+1)p+3\log k} < \alpha^{5kp}. \end{aligned}$$

In the above, we used the identity

$$F_1 + F_2 + \cdots + F_k = F_{k+2} - 1 < F_{k+2}.$$

Since $1/\log \alpha \in (2, 3)$, it follows that

$$kp/3 < 3M \quad \text{and} \quad 2M < 5kp,$$

which gives the desired bounds for kp versus M. Similarly,

$$(\alpha^{n/3})^q \leq (\alpha^{n-2})^q < F_n^q = M < (\alpha^{n-1})^q < \alpha^{nq},$$

so $nq/3 < 3M$ and $2M < nq$, which gives the desired bounds for nq versus M. $\square$

5. The linear forms

Our proof exploits five linear forms in logarithms together with their lower bounds (by linear forms in logarithms) provided that they are nonzero:

(7)
$$\begin{array}{rclrcl}
\Gamma_1 & := & kF_k^pF_n^{-q}-1; & \log|\Gamma_1| & > & -c_1kn(\log 10X)^2;\\
\Gamma_2 & := & kF_k^p\alpha^{-nq}\sqrt{5}^q-1; & \log\Gamma_2| & > & -c_1k(\log 10X)^2;\\
\Gamma_3 & := & \alpha^{p(k+1)}\left(\frac{k\alpha^p-(k+1)}{5^{p/2}(\alpha^p-1)^2}\right)F_n^{-q}-1; & \log|\Gamma_3| & > & -c_1pn(\log 10X)^2;\\
\Gamma_4 & := & \left(\frac{k\alpha^p-(k+1)}{(\alpha^p-1)^2}\right)5^{(q-p)/2}\alpha^{p(k+1)-qn}-1; & \log|\Gamma_4| & > & -c_1p(\log 10X)^2;\\
\Gamma_5 & := & k\alpha^{kp-qn}5^{(q-p)/2}-1; & \log|\Gamma_5| & > & -c_1(\log 10X)^2.
\end{array}$$

We will justify the fact that they are non-zero later. For now, let us assume that they are. The inequalities above follow from Theorem 2.1. It remains to assign some value for c_1. Well, each of the forms above is of the form

$$\Gamma := \gamma_1^{b_1}\cdots\gamma_t^{b_t}-1.$$

We explain the choices of t, $\gamma_1,\ldots,\gamma_t,b_1,\ldots,b_t$ for each of the above forms as well as upper bounds on the heights of the numbers $\gamma_1,\ldots,\gamma_t$. For Γ_1, we have

$$t:=3,\ \gamma_1:=k,\ \gamma_2:=F_k,\ \gamma_3:=F_n,\ b_1:=1,\ b_2:=p,\ b_3:=-q.$$

Then

$$B:=\max\{|b_i|\}<10X$$

by Lemma 4.1. This inequality is true for all of the above five forms except for Γ_3 and Γ_4, where $p(k+1)\le 2pk<20X$. Furthermore,

$$h(\gamma_1)=\log k<\log(10X),\ h(\gamma_2)=\log F_k<0.5k,\ h(\gamma_3)=\log F_n<0.5n.$$

For Γ_2, we have $t:=4$ and

$$\gamma_1:=k,\ \gamma_2:=F_k,\ \gamma_3:=\alpha,\ \gamma_4:=\sqrt{5},\ b_1:=1,\ b_2:=p,\ b_3:=-nq,\ b_4:=q.$$

Here, $h(\gamma_1)<\log(10X)$, $h(\gamma_2)<0.5k$, $h(\gamma_3)<1$, $h(\gamma_4)<1$. For Γ_3, we have $t:=3$ and

$$\gamma_1:=\alpha,\ \gamma_2:=\frac{k\alpha^p-(k+1)}{5^{p/2}(\alpha^p-1)^2},\ \gamma_3:=F_n,\ b_1:=p(k+1),\ \gamma_2:=1,\ \gamma_3:=-q.$$

Here, as we said, we take $B:=20X$, and we have $h(\gamma_1)<1$, $h(\gamma_3)<0.5n$. The only one that needs justification is the bound on $h(\gamma_2)$ which is, by the properties of the heights (1),

$$\begin{array}{rcl}
h(\gamma_2) & \le & h(k\alpha^p-(k+1))+2h(\alpha^p-1)+h(5^{p/2})\\
& \le & h(k)+ph(\alpha)+h(k+1)+2h(\alpha^p)+ph(5^{1/2})+3\log 2\\
& \le & (3\log\alpha/2)p+2\log k+4\log 2\le 10p(\log 10X).
\end{array}$$

For Γ_4, we have $t:=3$ and

$$\gamma_1:=\frac{k\alpha^p-(k+1)}{(\alpha^p-1)^2},\ \gamma_2:=\sqrt{5},\ \gamma_3:=\alpha,\ b_1:=1,\ b_2:=q-p,\ b_3:=p(k+1)-qn,$$

and for Γ_5, we have

$$t:=3,\ \gamma_1:=k,\ \gamma_2:=\alpha\ \gamma_3:=\sqrt{5},\ b_1:=1,\ b_2:=kp-qn,\ b_3:=q-p.$$

All actions are happening inside the quadratic field $\mathbb{K} := \mathbb{Q}[\sqrt{5}]$, which has degree $D = 2$. If the linear form involves some $\gamma \in \{\alpha, \sqrt{5}\}$ (numbers of heights smaller than 1), we then take the corresponding A to be 2. If the form involves $\gamma = k$, we take the corresponding A to be $2\log(10X)$. If the form involves F_k and F_n, we take the corresponding A to be k and n, respectively. Finally, if the form involves the number γ_2 of Γ_3, we take A to be $20p\log(10X)$. As for B, we have that the inequality $1 + \log B < 1 + \log(20X) < 2\log(10X)$ works for all five forms. It then follows that a uniform c_1 that works for all forms can be taken to be, by Theorem 2.1, an upper bound for

$$1.4 \times 30^{4+3} \times 4^{4.5} \times 2^2 \times (1 + \log 2) \times 2 \times 20 \times 2^2,$$

so we take $c_1 := 10^{17}$. We record this as a lemma.

Lemma 5.1. *For any $i = 1, \ldots, 5$, the inequality shown in Table (7) holds for Γ_i with $c_1 := 10^{17}$ assuming that $\Gamma_i \neq 0$.*

6. The nonvanishing of the first 4 forms

It is easy to see that $\Gamma_1,\ \Gamma_2$ are nonzero. Indeed, since

$$\sum_{j=1}^{k-1} jF_j^k = -(kF_k^p - F_n^q)$$

and $k > 1$, it follows that $\Gamma_1 < 0$. If $\Gamma_2 = 0$, then $\alpha^{2nq} = (kF_k^p)^2 5^{-q} \in \mathbb{Q}$, a contradiction.

For Γ_3, assuming that it is zero, we get

$$k\alpha^p - (k+1) = F_n^q \Delta_p 5^{p/2} \alpha^{-p(k+1)}.$$

Taking norms in $\mathbb{Q}[\sqrt{5}]$, we get

$$|((-1)^p - L_p + 1)k^2 + (2 - L_p)k + 1| = F_n^{2q} 5^p |(-1)^p - L_p + 1|.$$

For p even, we get

$$|(L_p - 2)k^2 + (L_p - 2)k - 1| = F_n^{2q} 5^p |L_p - 2|. \tag{8}$$

The case $L_p - 2 = 0$ is impossible since it leads to $1 = 0$. Thus, both sides above are nonzero. If p is odd, we get

$$|L_p k^2 + (L_p - 2)k - 1| = F_n^{2q} 5^p L_p^2, \tag{9}$$

and again both sides are nonzero. We thus get from (8) and (9) according to whether p is even or odd, respectively, that

$$\begin{aligned} X &< 10nq = \left(\frac{30}{\log \alpha}\right) q(n/3) \log \alpha < 100q(n-2)\log \alpha < \log F_n^{2q} \\ &< |\log(L_p k^2)| \leq \log L_p + 2\log k < p + 2\log k \\ &< X\left(\frac{10}{k} + \frac{2\log(10X)}{X}\right), \end{aligned}$$

so

$$1 < \frac{10}{k} + \frac{2\log(10X)}{X}.$$

Assuming that $X > 10^{10}$, it follows that

$$k \leq 10. \tag{10}$$

Finally, if $\Gamma_4 = 0$, then

$$k\alpha^p - (k+1) = 5^{(p-q)/2}\Delta_p^2\alpha^{qn-p(k+1)},$$

which upon taking norms becomes

$$|(-1)^p - L_p + 1)k^2 + (2 - L_p)k + 1| = 5^{p-q}((-1)^p - L_p + 1)^2.$$

These equations have been solved in [4] and they have no solutions with $k \geq 3$. So, we record what we have proved.

Lemma 6.1. *If $k \geq 3$, the forms $\Gamma_1,\ \Gamma_2,\ \Gamma_4$ are nonzero. Furthermore, if $X > 10^{10}$ and $\Gamma_3 = 0$, then $k \leq 10$.*

7. The terminology

Definition. *We say that*

$$F_m^j = \left(\frac{\alpha^m}{\sqrt{5}}\right)^j \left(1 + \frac{(-1)^m}{\alpha^{2m}}\right)^j$$

is "expandable", if the right parenthesis above satisfies

$$\left(1 + \frac{(-1)^m}{\alpha^{2m}}\right)^j = 1 + \zeta_{j,m}, \quad \textit{with} \quad |\zeta_{j,m}| < \frac{1}{\alpha^{1.5m}}. \tag{11}$$

Lemma 7.1. *If $\kappa > 0$ is any constant and $j < m^\kappa$, then estimate (11) above holds provided $m > c_2 := c_2(\kappa)$, where $c_2(\kappa)$ is such that*

$$2m^\kappa < \alpha^{0.5m} \quad \textit{holds for all} \quad m > c_2(\kappa).$$

Proof. Suppose that m is even. Then

$$\left(1 + \frac{(-1)^m}{\alpha^{2m}}\right)^j < \left(1 + \frac{1}{\alpha^{2m}}\right)^j < \exp\left(\frac{j}{\alpha^{2m}}\right) < 1 + \frac{2j}{\alpha^{2m}},$$

provided $j/\alpha^{2m} < 1/2$. Thus, we need $2j/\alpha^{2m} < 1/\alpha^{1.5}m$, or $2j < \alpha^{0.5m}$, so $2m^\kappa < \alpha^{0.5m}$, which clearly holds for $m > c_2(\kappa)$. A similar argument works for m odd. $\square$

8. The strategy

We will start with the linear form Γ_1 which is small. We look for positive constants $\kappa_1, \kappa_2, \kappa_3$ all smaller than 1 such that

$$k > X^{\kappa_1}, \quad p > X^{\kappa_2}, \quad n > X^{\kappa_3},$$

for $X > X_0$. For this subsection we shall ignore the implied constants (but we will make them explicit when time comes) and we use the Vinogradov symbols $\gg$ and $\ll$. If we succeed say first that $k > X^{\kappa_1}$, it then follows, by Lemma 4.1, that $p \ll X/k \ll X^{1-\kappa_1} \ll k^{(1-\kappa_1)/\kappa_1}$ so, by Lemma 7.1, F_k^p is expandable. This allows us to transform first Γ_1 into Γ_3. If this further allows us to conclude that also $n > X^{\kappa_3}$, then the same argument based on Lemma 4.1 and Lemma 7.1 shows that $q \ll X/n \ll X^{1-\kappa_3} \ll n^{(1-\kappa_3)/\kappa_3}$, which allows us to further transform Γ_1 into Γ_2 and Γ_3 into Γ_4. At each stage we need to worry about whether $\Gamma_i \neq 0$, but we already showed that this is the case for $i = 1, 2, 4$ and if $\Gamma_3 = 0$, then $X^{\kappa_1} < k \leq 10$ already gives a bound on X. At each stage, we will have bounds of the form

$$\log|\Gamma_i| \ll -a, \tag{12}$$

where a is one of the variables k, n, p, q. Comparing this with the corresponding lower bound on $\log|\Gamma_i|$ from the ith row and right–most column of Table 7, we get some upper bound on a in terms of some of the variables k, n, p, q. The goal is to recursively get to Γ_5 which will then imply that $a \ll (\log 10X)^2$. Assuming that this last a satisfies $a = \min\{p, k, n\}$, then $a > X^{\kappa_4}$ with $\kappa_4 := \min\{\kappa_1, \kappa_2, \kappa_3\}$, so we get a bound on X. There is still the possibility that Γ_5 is zero and taking care of this eventuality requires quite a bit of work. To deal with this last situation, we show that it leads to yet another (a sixth) linear form in logarithms of algebraic numbers which is small and non-zero, so we can apply Theorem 2.1 to it and get a bound on X. So, now the plan is clear, let's see the details.

9. The value of κ_1

Lemma 9.1. *We can take $\kappa_1 = 1/4$ for $X > 10^{200}$.*

Proof. We write X_0 for some number increasing from one iteration to the next which is a lower bound on X resulting from some inequality. At the end we collect the largest X_0 that we have encountered along the way. We assume that $k < X^{1/4}$ and we search for an acceptable X_0. From Lemma 4 in [4], we have that $F_j/F_{j+1} \le 2/3$ for all $j \ge 2$. In particular, $F_{k-j}/F_k \le (2/3)^j$ for all $j = 1, 2, \ldots, k-2$, while for $j = k-1$ this must be replaced by $F_1/F_k \le (2/3)^{k-2}$. Thus, in the left-hand side of (3), we have

$$M = kF_k^p \left(1 + \sum_{j=1}^{k-1} \left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p\right) = kF_k^p \left(1 + O\left(\frac{1}{1.5^p}\right)\right). \tag{13}$$

The constant inside the O can be taken to be 6 since

$$\sum_{j=1}^{k-1} \left(\frac{k-j}{k}\right)\left(\frac{F_{k-j}}{F_k}\right)^p < 2\sum_{j\ge 1}\left(\frac{2}{3}\right)^{jp} \le \frac{2}{1.5^p}\sum_{j\ge 0}\left(\frac{2}{3}\right)^j = \frac{6}{1.5^p}.$$

Thus,

$$F_n^q = kF_k^p\left(1 + O\left(\frac{1}{1.5^p}\right)\right),$$

which leads to

$$|(kF_k^p)^{-1}F_n^q - 1| < \frac{6}{1.5^p}.$$

The linear form on the left–hand side is the same as Γ_1, except that the exponents are replaced by their negatives. This changes neither the conclusion that it is non-zero nor its lower bound from Table 7, so we get

$$\log|\Gamma_1| < -p\log 1.5 + \log 6,$$

which is inequality (12) with $i = 1$ and $a := p$. Comparing it with the right–most entry of the first row in Table 7, we get

$$p\log 1.5 - \log 6 < 10^{17} nk (\log 10X)^2.$$

Thus,

$$p < \frac{1}{\log 1.5}\left(10^{17}nk(\log 10X)^2 + \log 6\right) < 10^{18}nk(\log 10X)^2.$$

Since $n < 10X/q < 100kp/q$ by Lemma 4.1, we get $p < 10^{20}(k^2p)/q(\log 10X)^2$, which leads to

$$q < 10^{20}k^2(\log 10X)^2. \tag{14}$$

Since we are assuming that $k < X^{1/4}$, we get $q < 10^{20}X^{1/2}(\log 10X)^2$. Thus,

$$n > 10^{-1}X/q > 10^{-21}X^{1/2}/(\log 10X)^2 > X^{1/3} \quad \text{provided} \quad X > X_0 := 10^{160}.$$

Thus, $q < 10X/n < 10X^{2/3} \le 10n^2 < n^3$. Since $n > X^{1/3} > 10$ we have that $q < n^3$. Lemma 7.1 shows that F_n^q is expandable for $n > n_0$, where we can take n_0 such that

$$2n^3 < \alpha^{0.5n} \quad \text{holds for all} \quad n > n_0.$$

We can take $n_0 := 100$, and for us $n > X^{1/3} > n_0$ holds for $X > X_0$. Thus, for $X > X_0$, we have

$$M = F_n^q = \frac{\alpha^{nq}}{5^{q/2}}\left(1 + O\left(\frac{1}{\alpha^n}\right)\right),$$

and the constant inside the O can be taken to be 1. Hence,

$$kF_k^p\left(1 + O\left(\frac{1}{1.5^p}\right)\right) = M = \frac{\alpha^{nq}}{5^{q/2}}\left(1 + O\left(\frac{1}{\alpha^n}\right)\right). \tag{15}$$

The constants in both O above can be taken to be 6. Note that $p > 100$. Indeed, if not then

$$100 > p > 10^{-1}X/k > 10^{-1}X^{3/4},$$

which gives $X < 10^4$, a contradiction. Since p and n are both large, both terms involving O are at most $1/2$ in absolute values in (15) so the cofactors of kF_k^p and $\alpha^{nq}/5^{q/2}$ are all in $[1/2, 3/2]$. Thus, the ratio of these two numbers is in $[1/3, 3]$. After some rearranging and using the fact that $\alpha > 1.5$, it leads to

$$|kF_k^p\alpha^{-nq}5^{q/2} - 1| < \frac{24}{1.5^{\min\{p,n\}}}.$$

We recognise that the above inequality is

$$\log|\Gamma_2| \ll -\min\{p,n\},$$

which is estimate (12) with $i = 2$ and $a := \min\{p,n\}$. Thus, by looking at the right–most column in the second row of the Table 7, we get

$$-\min\{p,n\}\log(1.5) + \log 24 < 10^{17}k(\log 10X)^2,$$

so

$$a < \frac{1}{\log 1.5}\left(10^{17}k(\log 10X)^2 + \log 24\right) < 10^{18}k(\log 10X)^2. \tag{16}$$

Assume that $a = n$. Then

$$n < 10^{18}k(\log 10X)^2. \tag{17}$$

Multiplying (14) and (17), we get

$$10^{-1}X < nq < 10^{38}k^3(\log 10X)^4 < 10^{38}X^{3/4}(\log 10X)^4.$$

Thus,

$$X^{1/4} < 10^{39}(\log 10X)^4, \quad \text{so} \quad X < X_0 := 10^{200}.$$

Assume next that $a = p$. Then estimate (16) tells us that

$$p < 10^{18}k(\log X)^2.$$

Since $p > 10^{-1}X/k$ by Lemma 4.1, we get

$$10^{-1}X/k < p < 10^{18}k(\log 10X)^2, \quad \text{therefore} \quad k > 10^{-10}X^{1/2}/(\log 10X).$$

Since $k < X^{1/4}$, we get $X^{1/4} < 10^{10}(\log 10X)$, which leads to $X < 10^{50}$, a contradiction. This finishes the proof of this lemma. □

10. The value of κ_2

Lemma 10.1. *We can take $\kappa_2 = 1/9$ for $X > 10^{2000}$.*

Proof. We follow the same convention about X_0 as in Lemma 9.1. We assume that $p < X^{1/9}$. We use Lemma 9.1 and Lemma 7.1 to conclude that F_j^p is expandable for all $j \geq \ell := \lfloor k/2 \rfloor + 1 (\geq 2)$. Indeed, for us,

$$p < 10X/k < 10k^3 = 80(k/2)^3 < (k/2)^4,$$

since $k/2 > (1/2)X^{1/4} > 80$ for $X > 10^{200}$. Thus, it suffices, by Lemma 7.1, that $\kappa/2 > c_2(4)$, where $c_2(4)$ is such that

$$2n^4 < \alpha^{0.5n} \quad \text{holds for all} \quad n > c_2(4),$$

and we can take $c_2(4) := 100$, which is acceptable since $k/2 > 0.5X_0^{1/4} > c_2(4)$. Thus,

$$F_j^p = \frac{\alpha^{pj}}{5^{p/2}}\left(1 + O\left(\frac{1}{\alpha^{k/2}}\right)\right) \quad \text{for} \quad j \in [\ell, k].$$

The constant inside O can be taken to be 1. Hence,

$$M = \sum_{j \leq \ell - 1} jF_j^p + \sum_{j=\ell}^{k} jF_j^p =: S_1 + S_2\left(1 + \frac{\zeta}{\alpha^{k/2}}\right), \quad |\zeta| < 1.$$

Note that

$$\begin{aligned} S_1 &= \ell F_{\ell-1}^p \left(1 + \sum_{j=1}^{\ell-2}\left(\frac{\ell-1-j}{\ell-1}\right)\left(\frac{F_{\ell-1-j}}{F_{\ell-1}}\right)^p\right) \\ &< 7kF_{\ell-1}^p < 7k\alpha^{(\ell-2)p} \leq 7k\alpha^{(k-2)p/2} = 7\sqrt{k}(kF_k^p)^{1/2} < 7(kM)^{1/2} < \frac{10M}{\alpha^{k/2}}. \end{aligned}$$

The above inequality follows since $M > kF_k > \alpha^{-2}(k\alpha^k) > (0.7)^2(k\alpha^k)$. Furthermore,

$$\begin{aligned} S_2 &= \sum_{j=\ell}^{k} \frac{\alpha^{jp}}{5^{p/2}} = \frac{1}{5^{p/2}}\sum_{j=\ell}^{k}\alpha^{jp} \\ &= \frac{\alpha^p}{5^{p/2}}\left(\sum_{j=1}^{k} j(\alpha^p)^{j-1} - \sum_{j=1}^{\ell-1} j(\alpha^p)^{j-1}\right) \\ &= \frac{\alpha^p}{5^{p/2}}\left(\frac{k\alpha^{(k+1)p} - (k+1)\alpha^{kp} + 1}{(\alpha^p-1)^2} - \frac{(\ell-1)\alpha^{\ell p} - \ell\alpha^{(\ell-1)p} + 1}{(\alpha^p-1)^2}\right) \\ &= \frac{\alpha^p}{5^{p/2}}\left(\frac{k\alpha^{(k+1)p} - (k+1)\alpha^{kp}}{(\alpha^p-1)^2}\right)\left(1 + O\left(\frac{1}{\alpha^{kp}} + \frac{1}{\alpha^{(k-\ell)p}}\right)\right) \\ &= \frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p-1)^2}\left(1 + O\left(\frac{1}{\alpha^{k/2}}\right)\right). \end{aligned} \tag{18}$$

In the above, we used that

$$k\alpha^{(k+1)p} - (k+1)\alpha^{kp} = k\alpha^{kp}(\alpha^p - (k+1)/k) \geq k\alpha^{kp}(\alpha - 4/3) > (k/4)\alpha^{kp}$$

for all $p \geq 1$ and $k \geq 3$. Thus, since $|(\ell-1)\alpha^{\ell p} - \ell\alpha^{(\ell-1)p} + 1| \leq \ell\alpha^{\ell p}$, it follows that the constant inside the first O in (18) can be taken to be 4. Thus, the constant in the second O can be taken to be 10. Hence,

$$S_2 = \frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\left(1 + O\left(\frac{1}{\alpha^{k/2}}\right)\right).$$

It thus follows that

$$M = O\left(\frac{M}{\alpha^{k/2}}\right) + \frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\left(1 + O\left(\frac{1}{\alpha^{k/2}}\right)\right),$$

where the constant in both O can be taken to be 10. Since k is large it is now clear that the first term above can also be absorbed into the error term. Thus,

$$M = \frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\left(1 + O\left(\frac{1}{\alpha^{k/2}}\right)\right), \tag{19}$$

where the constant in the last O can be taken to be 30. Consequently, our equation (3) is

$$\frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\left(1 + O\left(\frac{1}{\alpha^{k/2}}\right)\right) = F_n^q.$$

Since the factor involving O is in $(1/2, 3/2)$, by rearranging, it gives

$$\left|\alpha^{(k+1)p}\left(\frac{(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\right)F_n^{-q} - 1\right| = O\left(\frac{1}{\alpha^{k/2}}\right), \tag{20}$$

where the constant in the last O can be taken to be 100. In the left–hand side of (20), we recognise Γ_3, which we may assume that it is nonzero by Lemma 6.1 for $X > X_0$ since $k > X^{1/4} > 10$. Thus,

$$\log|\Gamma_3| \ll -k,$$

so we have inequality (12) for $i = 3$ and $a := k$. Looking in the appropriate entry in the Table 7, we get from estimate (20) and the fact that the constant in its O can be taken to be 100 that

$$(k/2)\log\alpha - \log 100 < 10^{17}pn(\log 10X)^2. \tag{21}$$

Thus,

$$k < \frac{2}{\log\alpha}\left(10^{17}pn(\log 10X)^2 + \log 100\right) < 10^{18}pn(\log 10X)^2.$$

Since $p < X^{1/9}$ and $k > X^{1/4}$, it follows that

$$n > 10^{-18}(k/(p(\log 10X)^2) > 10^{-18}X^{1/4-1/9}/(\log 10X)^2 > X^{1/8}, \quad X > X_0 := 10^{2000}.$$

We now use Lemma 7.1 to conclude that F_n^q is expandable. That is,

$$q < 10X/n < 10n^7 < n^8 \quad \text{since} \quad n > X_0^{1/8} > 10,$$

so we want that $n > c_2(8)$, where $c_2(8)$ is such that

$$2n^8 < \alpha^{0.5n} \quad \text{holds for} \quad n > c_2(8).$$

Since we can take $c_2(8) := 100$ and $n > 100$, we conclude that indeed F_n^q is expandable. Thus,

$$M = \frac{\alpha^{(k+1)p}(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\left(1 + O\left(\frac{1}{\alpha^{k/2}}\right)\right) = \frac{\alpha^{nq}}{5^{q/2}}\left(1 + O\left(\frac{1}{\alpha^n}\right)\right).$$

The constant in both O can be taken to be 30. Rearranging the above equality we get

$$\left|\alpha^{p(k+1)-qn}\left(\frac{(k\alpha^p - (k+1))}{5^{p/2}(\alpha^p - 1)^2}\right)5^{(q-p)/2} - 1\right| = O\left(\frac{1}{\alpha^{\min\{k/2,n\}}}\right), \tag{22}$$

where the constant in the last O can be taken to be 100. In the left–hand side, we recognise Γ_4, which we proved that it is nonzero. Thus, this gives

$$\log|\Gamma_4| \ll -\min\{k/2, n\}.$$

This is inequality (12) with $i = 4$ and $a := \min\{k/2, n\}$. Let us put some numbers in it. Looking at the appropriate entry in Table 7, we get

$$a\log\alpha - \log 100 < 10^{17}p(\log 10X)^2. \tag{23}$$

This implies

$$a < \frac{1}{\log\alpha}\left(10^{17}p(\log 10X)^2 + \log 100\right) < 10^{18}p(\log 10X)^2. \tag{24}$$

If $a = k/2$, then

$$k < 2 \times 10^{18}p(\log 10X)^2 < 10^{19}p(\log 10X)^2.$$

Since $k > 10^{-1}X/p$ by Lemma 4.1, we get

$$X/p < 10^{20}p(\log 10X)^2, \quad \text{therefore} \quad p > 10^{-10}X^{1/2}/(\log 10X).$$

Since we assumed that $p < X^{1/9}$, we get $X^{1/9} > 10^{-10}X^{1/2}/(\log 10X)$, which implies $X < 10^{50}$, which is false. If $a = n$, we get

$$n < 10^{18}p(\log 10X)^2. \tag{25}$$

Inserting (25) into (21) and using Lemma 9.1, we get

$$X^{1/4} < k < 10^{18}pn(\log 10X)^2 < 10^{36}p^2(\log 10X)^4 < 10^{36}X^{2/9}(\log 10X)^4,$$

which gives $X < 10^{1900}$, a contradiction. This finishes the proof of this lemma. □

11. The value of κ_3

Lemma 11.1. *We have $\kappa_3 = 1/10$ for $X > 10^{2499}$.*

Proof. We return to estimates to estimates (13), use Lemma 7.1 and the fact that F_k^p is expandable. Indeed, for this note that

$$p < 10X/k < 10k^3 < k^4,$$

and one checks that $k > c_2(4)$ in our range for $k > X_0^{1/4}$. Thus, we get that

$$\begin{aligned} M &= kF_k^p\left(1 + O\left(\frac{1}{1.5^p}\right)\right) = \frac{k\alpha^{pk}}{5^{p/2}}\left(1 + O\left(\frac{1}{\alpha^k}\right)\right)\left(1 + O\left(\frac{1}{1.5^p}\right)\right) \\ &= \frac{k\alpha^{pk}}{5^{p/2}}\left(1 + O\left(\frac{1}{1.5^{\min\{p,k\}}}\right)\right). \end{aligned}$$

The constant inside the O can be taken to be 3. Thus, equation (3) is

$$\frac{k\alpha^{pk}}{5^{p/2}}\left(1+O\left(\frac{1}{1.5^{\min\{p,k\}}}\right)\right)=F_n^q, \tag{26}$$

and can be rewritten as

$$|k\alpha^{pk}5^{-p/2}F_n^{-q}-1|=O\left(\frac{1}{1.5^{\min\{p,k\}}}\right),$$

where the constant inside the last O can be taken to be 10. In the left, we recognise Γ_2. Since $\Gamma_2\neq 0$, we get

$$\log|\Gamma_2|\ll -\min\{p,k\},$$

which is inequality (12) with $a:=\min\{p,k\}$. Looking in the appropriate entry in Table 7, we get that

$$\log 1.5\min\{p,k\}-\log 10<10^{17}n(\log 10X)^2,$$

which implies, by arguments used before, that

$$\min\{p,k\}<10^{18}n(\log 10X)^2.$$

Since $\min\{p,k\}>X^{1/9}$, by Lemmas 9.1 and 10.1, we get

$$n>10^{-18}X^{1/9}/(\log 10X)^2>X^{1/10}\quad\text{for}\quad X>10^{2499}.$$

□

12. The case when $\Gamma_5\neq 0$

Lemma 12.1. *We have $X\leq 10^{2499}$ provided $\Gamma_5\neq 0$.*

Proof. Assume $X>10^{2499}$. By Lemma 11.1, F_n^q is also expandable. Indeed we can check easily that $q<10X/n<10n^9<n^{10}$ and $n>c_2(10)$ in our range. So, equation (26) is

$$\frac{k\alpha^{kp}}{5^{p/2}}\left(1+O\left(\frac{1}{1.5^{\min\{p,k\}}}\right)\right)=F_n^q=\frac{\alpha^{qn}}{5^{q/2}}\left(1+O\left(\frac{1}{\alpha^n}\right)\right).$$

The constants inside both O can be taken to be 10. This can be rearranged as

$$|k\alpha^{pk-qn}5^{(q-p)/2}-1|=O\left(\frac{1}{1.5^{\min\{k,n,p\}}}\right),$$

where the constant inside the last O can be taken to be 100. In the left–hand side, we recognise Γ_5 which we assume it is nonzero. Thus,

$$\log|\Gamma_5|\ll -\min\{k,n,p\}.$$

This is estimate (12) with $a:=\min\{k,p,n\}$. Looking at the appropriate entry in Table (7), we have

$$\log 1.5\min\{k,n,p\}-\log 100<10^{17}(\log 10X)^2,$$

which implies

$$\min\{k,n,p\}<10^{18}(\log 10X)^2.$$

Since the left–hand side above is $>X^{1/10}$ by Lemmas 9.1, 10.1 and 11.1, we get $X^{1/10}<10^{18}(\log 10X)^2$, which gives $X<10^{300}$ a contradiction with $X>10^{2499}$.

□

13. The case when $\Gamma_5 = 0$

Lemma 13.1. *If $\Gamma_5 = 0$, then $X \le 10^{2499}$.*

Before giving the proof, note that this finishes the proof of our theorem since by Lemma 4.1, we have $\max\{k, n, p, q\} < 10X < 10^{2500}$.

Proof. This is messy. We write $X_0 := 10^{2499}$. The condition that $\Gamma_5 = 0$ implies the equality $\alpha^{kp-nq} = (5^{(p-q)/2}/k)$. Squaring this we get $\alpha^{2(kp-nq)} = 5^{p-q}/k^2 \in \mathbb{Q}$. The only possibility is that both sides of the above equality are 1. Thus, $kp = nq$ and $k = 5^{(p-q)/2}$. In particular, $p > q$, $p - q$ is even and $p - q = O(\log(10X))$, where the constant in the O can be taken to be 10. We look at secondary terms. We have

$$\begin{aligned} M &= kF_k^p\left(1+\frac{k-1}{k}\left(\frac{F_{k-1}}{F_k}\right)^p + O\left(\left(\frac{F_{k-2}}{F_k}\right)^p\right)\right) \\ &= \frac{k\alpha^{kp}}{5^{p/2}}\left(1+\frac{(-1)^k}{\alpha^{2k}}\right)^p\left(1+\frac{k-1}{k\alpha^p}\left(1+O\left(\frac{1}{\alpha^{2k}}\right)\right)^p + O\left(\frac{1}{\alpha^{2p}}\right)\right) \\ &= \frac{k\alpha^{kp}}{5^{p/2}}\left(1+\frac{p(-1)^k}{\alpha^{2k}} + O\left(\frac{1}{\alpha^{3k}}\right)\right)\left(1+\frac{(k-1)}{k\alpha^p} + O\left(\frac{1}{\alpha^{p+1.5k}}+\frac{1}{\alpha^{2p}}\right)\right) \\ &= \frac{k\alpha^{pk}}{5^{p/2}}\left(1+\frac{p(-1)^k}{\alpha^{2k}} + \frac{k-1}{k\alpha^p} + O\left(\frac{1}{\alpha^{2p}}+\frac{1}{\alpha^{p+1.5k}}+\frac{1}{\alpha^{3k}}\right)\right). \end{aligned}$$

By previous arguments (see Lemma 9.1), the constant in the first O above can be taken to be 10. In the second row of the above estimates, the constants inside the O can again be taken to be 10 (here we use that $F_k > \alpha^{k-2} = \alpha\alpha^{k-3} > \alpha F_{k-2}$) as well as on the third row above, so in the last O they can be chosen to be 25. Working on the other side of the equation, we have

$$\begin{aligned} F_n^q &= \frac{\alpha^{nq}}{5^{q/2}}\left(1+\frac{(-1)^n}{\alpha^{2n}}\right) \\ &= \frac{\alpha^{nq}}{5^{q/2}}\left(1+\frac{(-1)^n q}{\alpha^{2n}} + O\left(\frac{1}{\alpha^{3n}}\right)\right), \end{aligned}$$

where the constant inside this O can be chosen to be 10 as well. Since we have that $k\alpha^{kp}/5^{p/2} = \alpha^{nq}/5^{q/2}$, we may cancel these multiplicative factors and get

$$\frac{p(-1)^k}{\alpha^{2k}} + \frac{(k-1)}{k\alpha^p} - \frac{q(-1)^n}{\alpha^{2n}} = O\left(\frac{1}{\alpha^{2p}}+\frac{1}{\alpha^{p+1.5k}}+\frac{1}{\alpha^{3n}}+\frac{1}{\alpha^{3k}}\right). \tag{27}$$

We need to look at the spacing between p, $2k$, $2n$. The three terms above are all of the form $\exp(O(\log 10X))/\alpha^t$, where $t \in \{p, 2k, 2n\}$ and the constant inside this O is 1. Let $a := \min\{p, 2k, 2n\}$ and let $b \ge a$ be the next number in $\{p, 2k, 2n\}$. If $b - a > C_1 \log X$ with a sufficiently large constant C_1, we get a contradiction. Let us compute C_1. Say $a = p$. We then get

$$\frac{k-1}{k\alpha^p} \le \exp((\log 10X)\left(\frac{1}{\alpha^{2n}}+\frac{1}{\alpha^{2k}}\right) + O\left(\frac{1}{\alpha^{2p}}+\frac{1}{\alpha^{p+1.5k}}+\frac{1}{\alpha^{3n}}+\frac{1}{\alpha^{3k}}\right).$$

Multiplying by α^p, we get

$$
\begin{aligned}
(28)\qquad \frac{2}{3} &\le \exp((\log 10X))\left(\frac{1}{\alpha^{2n-p}}+\frac{1}{\alpha^{2k-p}}\right)\\
&+ O\left(\frac{1}{\alpha^p}+\frac{1}{\alpha^k}+\frac{1}{\alpha^{(2n-p)+n}}\right).
\end{aligned}
$$

Let C_1 be the constant implied by the above O. Assume that p is in such a way that $2n-p>3C_1\log X$ and $2k-p>3C_1\log X$. Then the first term above is

$$
<\frac{20X}{\alpha^{C_1\log X}}=\frac{20}{X^{C_1(\log\alpha^3-1)}}<\frac{20}{X}<\frac{1}{4},
$$

for $X>X_0$ provided we choose $C_1:=3$, since $3(\log\alpha^3-1)>1$. The last term inside the O on the right–hand side of estimate (28) is even smaller (much smaller than $1/4$). The same argument works if $a=2k$ or $a=2n$. This shows that the smallest of $p,\ 2k,\ 2n$ must be at distance at most $3\log X<10\log(10X)$ of the next largest one.

We distinguish 3 cases:

Case 1. $p-2k=O(\log(10X))$. Since $p-q=O(\log(10X))$, we have that $q-2k=(q-p)+(p-2k)=O(\log X)$. The constant in the last O is 20. Thus,

$$
\begin{aligned}
2n &= \frac{pk}{q}=2k\frac{(2k+O(\log(10X)))}{2k+O(\log(10X))}\\
&= 2k\left(1+O\left(\frac{\log(10X)}{k}\right)\right)\left(1+O\left(\frac{\log(10X)}{k}\right)\right)^{-1}\\
&= 2k\left(1+O\left(\frac{\log(10X)}{k}\right)\right)\\
&= 2k+O(\log(10X)).
\end{aligned}
$$

The constant in the last O above can be taken to be 100. In the above, we used Lemma 9.1 to the effect that $k>X^{1/4}$ to conclude that $\log X/k<\log X/X^{1/4}$ is a very small number for $X>X_0$. Thus,

$$
p=2k+u,\quad q=2k+v,\quad 2n=2k+w,\qquad \max\{|u|,|v|,|w|\}=O(\log(10X)).
$$

The constant inside this O is 100. Since $2pk=2nq$, we get

$$
(2k+u)2k=(2k+v)(2k+w),\qquad \text{so}\qquad 2k(u-v-w)=vw.
$$

If $u-v-w\ne 0$, then $vw\ne 0$. In this case, $k\mid vw$ and $k\le|vw|$. Since $k>X^{1/4}$, it follows that $X^{1/4}<k\le|uv|<10^4(\log(10X))^2$, which is impossible for $X>X_0$. Thus, for $X>X_0$, we must have $u=v+w$ so $vw=0$, therefore $v=0$ or $w=0$. This shows that either $n=k$ or $q=2k$. If $n=k$, we get $p=q$, but in this case the left–hand side of (3) is

$$
>kF_k^p=kF_n^q>F_n^q,
$$

a contradiction. Thus, $q=2k$ and from $pk=qn$, we get $p=2n$. Further, $2n-2k=p-q=O(\log(10X))$, where again the constant inside O can be taken to be 100.

Case 2. $p - 2n = O(\log(10X))$. Since $p - q = O(\log(10X))$, we get that $q - 2n = O(\log(10X))$. The constant inside the last O is 20. Thus,

$$\begin{aligned} 2k &= \frac{(2n)q}{p} = 2n\frac{(2n + O(\log(10X)))}{2n + O(\log(10X))} \\ &= 2n\left(1 + O\left(\frac{\log(10X)}{n}\right)\right)\left(1 + O\left(\frac{\log(10X)}{n}\right)\right)^{-1} \\ &= 2n\left(1 + O\left(\frac{\log(10X)}{n}\right)\right) \\ &= 2n + O(\log(10X)). \end{aligned}$$

The constant inside the last O can be taken to be 100. So, $2n - 2k = O(\log(10X))$, which implies $p - 2k = O(\log X)$. The constant inside this last O can be taken to be 200. So, this case leads to the same conclusion as the previous one up to a slightly worse constant inside the O.

Case 3. $2n - 2k = O(\log(10X))$. In all Cases 1,2,3 we got that their hypothesis implied that $2n - 2k = O(\log(10X))$ with the constant inside the O being at most 200. In the first cases, we also know that $p - 2n = O(\log(10X))$ as well, but not necessarily in the third case. Well, let us show that this must be so in Case 3 as well. Assume $|p - 2n| > 1000 \log X$. Recall that we cannot have that p is the smallest of all three numbers in this case. So, in fact it must be the case that p is the largest of $2k,\ 2n,\ p$. In the left–hand side of equation (27), we move the term corresponding to p to the right–hand side. Assuming say $k < n$, we get

$$\frac{p(-1)^k\alpha^{2n-2k} - q(-1)^n}{\alpha^{2n}} = \frac{-(k-1)}{k\alpha^p} + O\left(\frac{1}{\alpha^{2p}} + \frac{1}{\alpha^{3.5k}} + \frac{1}{\alpha^{3n}} + \frac{1}{\alpha^{3k}}\right). \tag{29}$$

As we said, the constant inside the above O can be taken to be 25. The number in the left–hand side is not zero since α^{2n-2k} is not rational. The numerator is a quadratic integer in $\mathbb{Q}[\sqrt{5}]$ so its norm is ≥ 1. Thus,

$$|(-1)^k p\alpha^{2n-2k} - q(-1)^n||(-1)^k p\beta^{2n-2k} - q| \geq 1.$$

This shows that

$$|(-1)^k p\alpha^{2n-2k} - q(-1)^n| > \exp(-\log(10X)).$$

In the above, we used that

$$|(-1)^k p\beta^{2n-2k} - q| < p + q < 10X(1/k + 1/n) < 10X.$$

With the previous argument, it follows that if the inequality $p - 2n > 10\log(10X)$ holds, then we would get

$$\frac{1}{10X} < \frac{1}{(10X)^{10(\log\alpha^3 - 1)}} + O\left(\frac{1}{\alpha^{(p-2n)+p}} + \frac{1}{\alpha^{(2k-2n)+1.5k}} + \frac{1}{\alpha^n} + \frac{1}{\alpha^{k+(2k-2n)}}\right),$$

and $10(\log\alpha^3 - 1) > 4$, so the first term of the above inequality is $< 1/(20X)$. We thus get that

$$\frac{1}{20X} < 100\left(\frac{1}{\alpha^{(p-2n)+p}} + \frac{1}{\alpha^{(2k-2n)+1.5k}} + \frac{1}{\alpha^n} + \frac{1}{\alpha^{k+(2k-2n)}}\right),$$

Since $k > X^{1/4}$ and $|2n-2k| < 10\log(10X)$, we get that the minimum of the above exponents is at least as large as $\min\{k/2, n\} > X^{1/10}$, so we get

$$X^{1/10}\log\alpha < \log(800X),$$

which is of course false for $X > X_0$. The same argument works when $k > n$. Thus, it must be the case that even when the first (smallest) two numbers among $\{p, 2n, 2k\}$ are $2k$ and $2n$, we also have $p - 2n = O(\log X)$, where the constant inside this last O can be taken to be 1000. Up to the constant inside the O, this is in fact one of the Cases 1 or 2. Hence, all assumptions and conclusions from Cases 1, 2, 3 simultaneously hold with the constant inside all O being 1000, and in this case we saw that we must have $p = 2n$, $2n - 2k = p - q > 0$ for $X > X_0$. In this final situation, the left–hand side of equation (27) is

$$\frac{(k-1)/k + \alpha^{p-2n}((-1)^k(p\alpha^{2n-2k} - q(-1)^n)}{\alpha^p}.$$

and the above expression is nonzero since $(k-1)/k$ is not an algebraic integer. Thus,

$$\begin{aligned} & (k-1)/k\left|1-(-1)^{k-1}(k/(k-1))^{-1}\alpha^{p-2n}(p(-1)^k\alpha^{2n-2k}-q(-1)^n)\right| \\ < \quad & 100\left(\frac{1}{\alpha^{p+O(\log(10X)}}+\frac{1}{\alpha^{k+O(\log(10X)}}+\frac{1}{\alpha^{n+O(\log(10X))}}\right). \end{aligned}$$

All the constants inside the three exponents above are at most 1000. We know that $\min\{p, k, n\} > X^{1/10}$. We need a lower bound on the left–hand side above. Here is our new linear forms in logarithms, namely Γ_6. It is a linear form in $t := 3$ logarithms with

$$(\gamma_1, \gamma_2, \gamma_3, b_1, b_2, b_3) := ((-1)^{k-1}k/(k-1), \alpha, (-1)^k p\alpha^{2n-2k} - q(-1)^n, -1, p-2n, 1).$$

It is not zero since $(k-1)/k$ is not an algebraic integer. Furthermore, we have that $h(\gamma_1) < \log k < \log(10X)$, $h(\gamma_2) < 0.5$ Clearly, we can take $B := 10X$. As for $h(\gamma_3)$, we have

$$h(\gamma_3) \le \log p + \log q + |p-2n|h(\alpha) + \log 2 < 10^3\log(10X).$$

Thus,

$$\log|\Gamma_6| > -c_3(\log(10X))^3,$$

where we can take c_3 an upper bound for

$$1.4 \times 30^{3+3} \times 3^{4.5} \times 2^2 \times (1 + \log 2) \times 2^2 \times 2 \times 10^3.$$

So, we can take $c_3 := c_1 = 10^{17}$, and we then get that

$$(\min\{p/2, k/2, n/2\} - 1000\log(10X))\log\alpha - \log(300) < 10^{17}(\log 10X))^3 + \log 2.$$

The last $\log 2$ comes as an upper bound of $\log(k/(k-1))$. Since we know that $\min\{p/2, k/2, n/2\} > 0.5X^{1/10}$, we get

$$0.5X^{1/10} < \frac{1}{\log\alpha}\left(10^{17}(\log 10X)^3 + \log(600)\right) + 1000\log(10X) < 5\times10^{17}(\log(10X))^3,$$

so $X^{1/10} < 10^{18}(\log(10X))^3$, which is impossible for $X > X_0$. This finishes the proof. $\square$

14. Comments on computations

We did not make any attempts to reduce the bounds. However, we believe new ideas will be needed to lower the bounds to the range where one can just enumerate the solutions. Indeed, in order to reduce the variables, in the most fortunate case where $\Gamma_5 \neq 0$, one is lead to a final inequality of the type

$$|x \log \alpha - y \log 5^{1/2} + \log k| \ll \exp\{-\delta z\}, \tag{30}$$

where

$$(x, y, z) := (kp - qn, q - p, \min\{k, p, q\}),$$

and $\delta > 0$ in (30) is some small number. At this step, one applies a reduction method due to Baker and Davenport which only requires bounds on the variables x and y and returns a reasonably small value on z. However, in order for that to work, k has be to be known. Our k is known in the sense that it is bounded but the bound on it is astronomical. Thus, one cannot hope to completely solve the given equation unless one can come up with some reasonable small bounds on k (say 10^{10}), which our current method cannot provide.

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